Almost sure Weyl asymptotics for non-self-adjoint elliptic operators on compact manifolds

William Bordeaux Montrieux* Johannes Sjöstrand†‡

Abstract

In this paper, we consider elliptic differential operators on compact manifolds with a random perturbation in the 0th order term and show under fairly weak additional assumptions that the large eigenvalues almost surely distribute according to the Weyl law, well-known in the self-adjoint case.

Résumé

Dans ce travail, nos considérons des opérateurs différentiels elliptiques sur des variétés compactes avec une perturbation aléatoire dans le terme d'orde 0. Sous des hypothèses supplémentaires assez faibles, nous montrons que les grandes valeurs propres se distribuent selon la loi de Weyl, bien connue dans le cas auto-adjoint.

Contents

*Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, 1090 Wien, Austria, william.bordeaux-montrieux@univie.ac.at

†IMB, Université de Bourgogne, 9, Av. A. Savary, BP 47870, FR 21078 Dijon cédex, France, and UMR 5584 CNRS, johannes.sjostrand@u-bourgogne.fr

‡Ce travail a bénéficié d'une aide de l'Agence Nationale de la Recherche portant la référence ANR-08-BLAN-0228-01

1 Introduction

This work is a continuation of a series of works concerning the asymptotic distribution of eigenvalues for non-self-adjoint (pseudo-)differential operators with random perturbations. Since the works of L.N. Trefethen [8], E.B. Davies [2], M. Zworski [9] and many others (see for instance [4] for further references) we know that the resolvents of such operators tend to have very large norms when the spectral parameter is in the range of the symbol, and consequently, the eigenvalues are unstable under small perturbations of the operator. It is therefore quite natural to study the effect of random perturbations. Mildred Hager [4] studied quite general classes of non-self-adjoint h-pseudodifferential operators on the real line with a suitable random potential added, and she showed that the eigenvalues distribute according to the natural Weyl law with a probability very close to 1 in the semi-classical limit ($h \to 0$). Due to the method, this result was restricted to the interior of the range of the leading symbol p of the operator and with a non-vanishing assumption on the Poisson bracket $\{p, \overline{p}\}$.

In [5] the results were generalized to higher dimension and the boundary of the range of p could be included, but the perturbations where no more multiplicative. In [6, 7] further improvements of the method were introduced and the case of multiplicative perturbations was handled in all dimensions.

W. Bordeaux Montrieux [1] studied elliptic systems of differential operators on S^1 with random perturbations of the coefficients, and under some additional assumptions, he showed that the large eigenvalues obey the Weyl law *almost surely.* His analysis was based on a reduction to the semi-classical case (using essentially the Borel-Cantelli lemma), where he could use and extend the methods of Hager [4].

The purpose of the present work is to extend the results of [1] to the case of elliptic operators on compact manifolds by replacing the one dimensional semi-classical techniques by the more recent result of [7]. For simplicity, we treat only the scalar case and the random perturbation is a potential.

Let X be a smooth compact manifold of dimension n. Let P^0 be an elliptic differential operator on X of order $m \geq 2$ with smooth coefficients and with principal symbol $p(x,\xi)$. In local coordinates we get, using standard multi-index notation,

$$P^0 = \sum_{|\alpha| \leq m} a_\alpha^0(x) D^\alpha, \quad p(x,\xi) = \sum_{|\alpha| = m} a_\alpha^0(x) \xi^\alpha. \tag{1.1}$$

Recall that the ellipticity of P^0 means that $p(x,\xi) \neq 0$ for $\xi \neq 0$. We assume that

$$p(T^*X) \neq \mathbf{C}. \tag{1.2}$$

Fix a strictly positive smooth density of integration dx on X, so that the L^2 norm $\|\cdot\|$ and inner product $(\cdot|\cdot\cdot)$ are unambiguously defined. Let $\Gamma : L^2(X) \to L^2(X)$ be the antilinear operator of complex conjugation, given by $\Gamma u = \overline{u}$. We need the symmetry assumption

$$P^* = \Gamma P \Gamma, \tag{1.3}$$

where P^* is the formal complex adjoint of P. As in [7] we observe that the property (1.3) implies that

$$p(x, -\xi) = p(x, \xi), \tag{1.4}$$

and conversely, if (1.4) holds, then the operator $\frac{1}{2}(P + \Gamma P \Gamma)$ has the same principal symbol p and satisfies (1.3).

Let $\widetilde{R}$ be an elliptic differential operator on X with smooth coefficients, which is self-adjoint and strictly positive. Let $\epsilon_0, \epsilon_1, ...$ be an orthonormal basis of eigenfunctions of $\widetilde{R}$ so that

$$\widetilde{R}\epsilon_j = (\mu_j^0)^2 \epsilon_j, \quad 0 < \mu_0^0 < \mu_1^0 \leq \mu_2^0 \leq ... \tag{1.5}$$

Our randomly perturbed operator is

$$P_\omega^0 = P + q_\omega^0(x), \tag{1.6}$$

where ω is the random parameter and

$$q_\omega^0(x) = \sum_0^\infty \alpha_j^0(\omega)\epsilon_j. \tag{1.7}$$

Here we assume that $\alpha_j^0(\omega)$ are independent complex Gaussian random variables of variance σ_j^2 and mean value 0:

$$\alpha_j^0 \sim \mathcal{N}(0, \sigma_j^2), \tag{1.8}$$

where

$$(\mu_j^0)^{-\rho} e^{-(\mu_j^0)^{\frac{\beta}{M+1}}} \lesssim \sigma_j \lesssim (\mu_j^0)^{-\rho}, \tag{1.9}$$

$$M = \frac{3n - \frac{1}{2}}{s - \frac{n}{2} - \epsilon},\ 0 \leq \beta < \frac{1}{2},\ \rho > n, \tag{1.10}$$

where s, ρ, ϵ are fixed constants such that

$$\frac{n}{2} < s < \rho - \frac{n}{2},\ 0 < \epsilon < s - \frac{n}{2}.$$

Let $H^s(X)$ be the standard Sobolev space of order s. As will follow from considerations below, we have $q_\omega^0 \in H^s(X)$ almost surely since $s < \rho - \frac{n}{2}$. Hence $q_\omega^0 \in L^\infty$ almost surely, implying that P_ω^0 has purely discrete spectrum.

Consider the function $F(\omega) = \arg p(\omega)$ on S^*X. For given $\theta_0 \in S^1 \simeq \mathbf{R}/(2\pi\mathbf{Z})$, $N_0 \in \dot{\mathbf{N}} := \mathbf{N} \setminus \{0\}$, we introduce the property $P(\theta_0, N_0)$:

$$\sum_1^{N_0} |\nabla^k F(\omega)| \neq 0 \text{ on } \{\omega \in S^*X;\, F(\omega) = \theta_0\}. \tag{1.11}$$

Notice that if $P(\theta_0, N_0)$ holds, then $P(\theta, N_0)$ holds for all θ in some neighborhood of θ_0.

We can now state our main result.

Theorem 1.1 *Assume that $m \geq 2$. Let $0 \leq \theta_1 \leq \theta_2 \leq 2\pi$ and assume that $P(\theta_1, N_0)$ and $P(\theta_2, N_0)$ hold for some $N_0 \in \dot{\mathbf{N}}$. Let $g \in C^\infty([\theta_1, \theta_2];]0, \infty[)$ and put*

$$\Gamma^g_{\theta_1,\theta_2;0,\lambda} = \{re^{i\theta}; \theta_1 \leq \theta \leq \theta_2,\ 0 \leq r \leq \lambda g(\theta)\}.$$

Then for every $\delta \in]0, \frac{1}{2} - \beta[$ there exists $C > 0$ such that almost surely: $\exists C(\omega) < \infty$ such that for all $\lambda \in [1, \infty[$:

$$\begin{aligned} |\#(\sigma(P_\omega^0) \cap \Gamma^g_{\theta_1,\theta_2;0,\lambda}) - \frac{1}{(2\pi)^n} \operatorname{vol} p^{-1}(\Gamma^g_{\theta_1,\theta_2;0,\lambda})| \\ \leq C(\omega) + C\lambda^{\frac{n}{m} - \frac{1}{m}(\frac{1}{2} - \beta - \delta)\frac{1}{N_0+1}}. \end{aligned} \tag{1.12}$$

Here $\sigma(P_\omega^0)$ denotes the spectrum and $\#(A)$ denotes the number of elements in the set A. In (1.12) the eigenvalues are counted with their algebraic multiplicity.

The proof actually allows to have almost surely a simultaneous conclusion for a whole family of θ_1, θ_2, g:

Theorem 1.2 *Assume that $m \geq 2$. Let Θ be a compact subset of $[0, 2\pi]$. Let $N_0 \in \mathbf{N}$ and assume that $P(\theta, N_0)$ holds uniformly for $\theta \in \Theta$. Let $\mathcal{G}$ be a subset of $\{(g, \theta_1, \theta_2);\ \theta_j \in \Theta, \theta_1 \leq \theta_2,\ g \in C^\infty([\theta_1, \theta_2];]0, \infty[)\}$ with the property that g and $1/g$ are uniformly bounded in $C^\infty([\theta_1, \theta_2];]0, \infty[)$ when (g, θ_1, θ_2) varies in $\mathcal{G}$. Then for every $\delta \in]0, \frac{1}{2} - \beta[$ there exists $C > 0$ such that almost surely: $\exists C(\omega) < \infty$ such that for all $\lambda \in [1, \infty[$ and all $(g, \theta_1, \theta_2) \in \mathcal{G}$, we have the estimate (1.12).*

The condition (1.9) allows us to choose σ_j decaying faster than any negative power of μ_j^0. Then from the discussion below, it will follow that $q_\omega(x)$

is almost surely a smooth function. A rough and somewhat intuitive interpretation of Theorem 1.2 is then that for almost every elliptic operator of order ≥ 2 with smooth coefficients on a compact manifold which satisfies the conditions (1.2), (1.3), the large eigenvalues distribute according to Weyl's law in sectors with limiting directions that satisfy a weak non-degeneracy condition.

2 Volume considerations

In the next section we shall perform a reduction to a semi-classical situation and work with $h^m P_0$ which has the semi-classical principal symbol p in (1.1). As in [5, 6, 7], we introduce

$$V_z(t) = \mathrm{vol}\,\{\rho \in T^*X;\ |p(\rho) - z|^2 \leq t\},\ t \geq 0. \tag{2.1}$$

Proposition 2.1 *For any compact set $K \subset \dot{\mathbf{C}} = \mathbf{C} \setminus \{0\}$, we have*

$$V_z(t) = \mathcal{O}(t^\kappa),\ \text{uniformly for } z \in K,\ 0 \leq t \ll 1, \tag{2.2}$$

with $\kappa = 1/2$.

The property (2.2) for some $\kappa \in]0,1[$ is required in [5, 6, 7] near the boundary of the set Γ, where we count the eigenvalues. Another important quantity appearing there was

$$\mathrm{vol}\,(\gamma + D(0,t)), \tag{2.3}$$

where $\gamma = \partial\Gamma$ and $\Gamma \Subset \dot{\mathbf{C}}$ is assumed to have piecewise smooth boundary. From (2.2) with general κ it follows that the volume (2.3) is $\mathcal{O}(t^{2\kappa-1})$, which is of interest when $\kappa > 1/2$. In our case, we shall therefore investigate $\mathrm{vol}\,(\gamma + B(0,t))$ more directly, when γ is (the image of) a smooth curve. The following result implies Proposition 2.1:

Proposition 2.2 *Let γ be the curve $\{re^{i\theta} \in \mathbf{C};\ r = g(\theta),\ \theta \in S^1\}$, where $0 < g \in C^1(S^1)$. Then*

$$\mathrm{vol}\,(p^{-1}(\gamma + D(0,t))) = \mathcal{O}(t),\ t \to 0.$$

Proof. This follows from the fact that the radial derivative of p is $\neq 0$. More precisely, write $T^*X \setminus 0 \ni \rho = r\omega,\ \omega \in S^*X,\ r > 0$, so that $p(\rho) = r^m p(\omega)$, $p(\omega) \neq 0$. If $\rho \in p^{-1}(\gamma + D(0,t))$, we have for some $C \geq 1$, independent of t,

$$g(\arg p(\omega)) - Ct \leq r^m |p(\omega)| \leq g(\arg p(\omega)) + Ct,$$

$$\left(\frac{g(\arg p(\omega)) - Ct}{|p(\omega)|}\right)^{\frac{1}{m}} \le r \le \left(\frac{g(\arg p(\omega)) + Ct}{|p(\omega)|}\right)^{\frac{1}{m}},$$

so for every $\omega \in S^*X$, r has to belong to an interval of length $\mathcal{O}(t)$. $\square$

We next study the volume in (2.3) when γ is a radial segment of the form $[r_1, r_2]e^{i\theta_0}$, where $0 < r_1 < r_2$ and $\theta_0 \in S^1$.

Proposition 2.3 *Let $\theta_0 \in S^1$, $N_0 \in \dot{\mathbf{N}}$ and assume that $P(\theta_0, N_0)$ holds. Then if $0 < r_1 < r_2$ and γ is the radial segment $[r_1, r_2]e^{i\theta_0}$, we have*

$$\mathrm{vol}\,(p^{-1}(\gamma + D(0,t))) = \mathcal{O}(t^{1/N_0}),\ t \to 0.$$

Proof. We first observe that it suffices to show that

$$\mathrm{vol}_{S^*X} F^{-1}([\theta_0 - t, \theta_0 + t]) = \mathcal{O}(t^{1/N_0}).$$

This in turn follows for instance from the Malgrange preparation theorem: At every point $\omega_0 \in F^{-1}(\theta_0)$ we can choose coordinates $\omega_1, ..., \omega_{2n-1}$, centered at ω_0, such that for some $k \in \{1, ..., N_0\}$, we have that $\partial_{\omega_1}^j (F - \theta_0)(\omega_0)$ is $= 0$ when $0 \le j \le k-1$ and $\neq 0$ when $j = k$. Then by Malgrange's preparation theorem, we have

$$F(\omega) - \theta_0 = G(\omega)(\omega_1^k + a_1(\omega_2, ..., \omega_{2n-1})\omega_1^{k-1} + ... + a_k(\omega_2, ..., \omega_{2n-1})),$$

where G, a_j are real and smooth, $G(\omega_0) \neq 0$, and it follows that

$$\mathrm{vol}\,(F^{-1}([\theta_0 - t, \theta_0 + t]) \cap \mathrm{neigh}\,(\omega_0)) = \mathcal{O}(t^{1/k}).$$

It then suffices to use a simple compactness argument. $\square$

Now, let $0 \le \theta_1 < \theta_2 \le 2\pi$, $g \in C^\infty([\theta_1, \theta_2];]0, \infty[)$ and put

$$\Gamma^g_{\theta_1,\theta_2;r_1,r_2} = \{re^{i\theta};\ \theta_1 \le \theta \le \theta_2,\ r_1 g(\theta) \le r \le r_2 g(\theta)\}, \tag{2.4}$$

for $0 \le r_1 \le r_2 < \infty$. If $0 < r_1 < r_2 < +\infty$ and $P(\theta_j, N_0)$ hold for $j = 1, 2$, then the last two propositions imply that

$$\mathrm{vol}\, p^{-1}(\partial \Gamma^g_{\theta_1,\theta_2;r_1,r_2} + D(0,t)) = \mathcal{O}(t^{1/N_0}),\ t \to 0. \tag{2.5}$$

3 Semiclassical reduction

We are interested in the distribution of large eigenvalues ζ of P^0_ω, so we make a standard reduction to a semi-classical problem by letting $0 < h \ll 1$ satisfy

$$\zeta = \frac{z}{h^m},\ |z| \asymp 1,\ h \asymp |\zeta|^{-1/m}, \tag{3.1}$$

and write

$$h^m(P^0_\omega - \zeta) = h^m P^0_\omega - z =: P + h^m q^0_\omega - z, \tag{3.2}$$

where

$$P = h^m P^0 = \sum_{|\alpha|\le m} a_\alpha(x;h)(hD)^\alpha. \tag{3.3}$$

Here

$$\begin{aligned} a_\alpha(x;h) &= \mathcal{O}(h^{m-|\alpha|}) \text{ in } C^\infty, \\ a_\alpha(x;h) &= a^0_\alpha(x) \text{ when } |\alpha| = m. \end{aligned} \tag{3.4}$$

So P is a standard semi-classical differential operator with semi-classical principal symbol $p(x,\xi)$.

Our strategy will be to decompose the random perturbation

$$h^m q^0_\omega = \delta Q_\omega + k_\omega(x),$$

where the two terms are independent, and with probability very close to 1, δQ_ω will be a semi-classical random perturbation as in [7] while

$$\|k_\omega\|_{H^s} \le h, \tag{3.5}$$

and

$$s \in]\frac{n}{2}, \rho - \frac{n}{2}[\tag{3.6}$$

is fixed. Then $h^m P^0_\omega$ will be viewed as a random perturbation of $h^m P^0 + k_\omega$. In order to achieve this without extra assumptions on the order m, we will also have to represent some of our eigenvalues $\alpha^0_j(\omega)$ as sums of two independent Gaussian random variables.

We start by examining when

$$\|h^m q^0_\omega\|_{H^s} \le h. \tag{3.7}$$

Proposition 3.1 *There is a constant $C > 0$ such that (3.7) holds with probability*

$$\ge 1 - \exp(C - \frac{1}{2Ch^{2(m-1)}}).$$

Proof. We have

$$h^m q^0_\omega = \sum_0^\infty \alpha_j(\omega)\epsilon_j, \quad \alpha_j = h^m \alpha^0_j \sim \mathcal{N}(0, (h^m\sigma_j)^2), \tag{3.8}$$

and the α_j are independent. Now, using standard functional calculus for $\widetilde{R}$ as in [6, 7], we see that

$$\|h^m q_\omega^0\|_{H^s}^2 \asymp \sum_0^\infty |(\mu_j^0)^s \alpha_j(\omega)|^2, \tag{3.9}$$

where $(\mu_j^0)^s \alpha_j \sim \mathcal{N}(0, (\widetilde{\sigma}_j)^2)$ are independent random variables and $\widetilde{\sigma}_j = (\mu_j^0)^s h^m \sigma_j$.

Now recall the following fact, established by Bordeaux Montrieux [1], improving and simplifying a similar result in [5]: Let $d_0, d_1, ...$ be a finite or infinite family of independent complex Gaussian random variables, $d_j \sim \mathcal{N}(0, (\widehat{\sigma}_j)^2)$, $0 < \widehat{\sigma}_j < \infty$, and assume that $\sum \widehat{\sigma}_j^2 < \infty$. Then for every $t > 0$,

$$\mathbf{P}(\sum |d_j|^2 \geq t) \leq \exp(\frac{-1}{2\max \widehat{\sigma}_j^2}(t - C_0 \sum \widehat{\sigma}_j^2)). \tag{3.10}$$

Here $\mathbf{P}(A)$ denotes the probability of the event A and $C_0 > 0$ is a universal constant. The estimate is interesting only when $t > C_0 \sum \widehat{\sigma}_j^2$ and for such values of t it improves if we replace $\{d_0, d_1, ...\}$ by a subfamily. Indeed, $\sum \widehat{\sigma}_j^2$ will then decrease and so will $\max \widehat{\sigma}_j^2$.

Apply this to (3.9) with $d_j = (\mu_j^0)^s \alpha_j$, $t = h^2$. Here, we recall that $\widetilde{\sigma}_j = (\mu_j^0)^s h^m \sigma_j$, and get from (1.9), (3.6) that

$$\max \widetilde{\sigma}_j^2 \asymp h^{2m}, \tag{3.11}$$

while

$$\sum_0^\infty \widetilde{\sigma}_j^2 \lesssim h^{2m} \sum_0^\infty (\mu_j^0)^{2(s-\rho)}. \tag{3.12}$$

Let $N(\mu) = \#(\sigma(\sqrt{\widetilde{R}}) \cap]0, \mu])$ be the number of eigenvalues of $\sqrt{\widetilde{R}}$ in $]0, \mu]$, so that $N(\mu) \asymp \mu^n$ by the standard Weyl asymptotics for positive elliptic operators on compact manifolds. The last sum in (3.12) is equal to

$$\int_0^\infty \mu^{2(s-\rho)} dN(\mu) = \int_0^\infty 2(\rho - s)\mu^{2(s-\rho)-1} N(\mu) d\mu,$$

which is finite since $2(s - \rho) + n < 0$ by (3.6). Thus

$$\sum_0^\infty \widetilde{\sigma}_j^2 \lesssim h^{2m}, \tag{3.13}$$

and the proposition follows from applying (3.9), (3.11), (3.12) to (3.10) with $t = h^2$. $\square$

We next review the choice of parameters for the random perturbation in [7] (and [6]). This perturbation is of the form δQ_ω,

$$Q_\omega = h^{N_1} q_\omega,\ \delta = \tau_0 h^{N_1+n},\ 0 < \tau_0 \le \sqrt{h}, \tag{3.14}$$

where

$$q_\omega(x) = \sum_{0<h\mu_k^0\le L} \alpha_k(\omega)\epsilon_k(x),\ |\alpha|_{\mathbf{C}^D} \le R, \tag{3.15}$$

and a possible choice of L, R is

$$L = Ch^{-M}, \quad R = Ch^{-\widetilde{M}}, \tag{3.16}$$

with

$$M = \frac{3n-\kappa}{s-\frac{n}{2}-\epsilon}, \quad \widetilde{M} = \frac{3n}{2} - \kappa + (\frac{n}{2}+\epsilon)M. \tag{3.17}$$

Here $\epsilon > 0$ is any fixed parameter in $]0, s-\frac{n}{2}[$ and $\kappa \in]0,1]$ is the geometric exponent appearing in (2.2), in our case equal to 1/2.

The exponent N_1 is given by

$$N_1 = \widetilde{M} + sM + \frac{n}{2}, \tag{3.18}$$

and q_ω should be subject to a probability density on $B_{\mathbf{C}^D}(0,R)$ of the form $C(h)e^{\Phi(\alpha;h)}L(d\alpha)$, where

$$|\nabla_\alpha \Phi| = \mathcal{O}(h^{-N_4}), \tag{3.19}$$

for some constant $N_4 \ge 0$.

Write

$$q_\omega^0 = q_\omega^1 + q_\omega^2, \tag{3.20}$$

$$q_\omega^1 = \sum_{0<h\mu_j^0\le L} \alpha_j^0(\omega)\epsilon_j,\ q_\omega^2 = \sum_{h\mu_j^0 > L} \alpha_j^0(\omega)\epsilon_j. \tag{3.21}$$

From Proposition 3.1 and its proof, especially the observation after (3.10), we know that

$$\|h^m q_\omega^2\|_{H^s} \le h \text{ with probability } \ge 1 - \exp(C_0 - \frac{1}{2Ch^{2(m-1)}}). \tag{3.22}$$

We write

$$P + h^m q_\omega^0 = (P + h^m q_\omega^2) + h^m q_\omega^1,$$

and recall that the main result in [7] is valid also when P is replaced by the perturbation $P + h^m q_\omega^2$, provided that $\|h^m q_\omega^2\|_{H^s} \le h$.

The next question is then wether $h^m q_\omega^1$ can be written as $\tau_0 h^{2N_1+n} q_\omega$ where $q_\omega = \sum_{0<h\mu_j^0\le L} \alpha_j\epsilon_j$ and $|\alpha|_{\mathbf{C}^D} \le R$ with probability close to 1. We get

$$\alpha_j = \frac{1}{\tau_0} h^{m-2N_1-n}\alpha_j^0(\omega) \sim \mathcal{N}(0,\widehat{\sigma}_j^2),$$

$$\frac{1}{\tau_0} h^{m-2N_1-n}(\mu_j^0)^{-\rho} e^{-(\mu_j^0)^{\frac{\beta}{M+1}}} \lesssim \widehat{\sigma}_j \lesssim \frac{1}{\tau_0} h^{m-2N_1-n}(\mu_j^0)^{-\rho}.$$

Applying (3.10), we get

$$\mathbf{P}(|\alpha|_{\mathbf{C}^D}^2 \ge R^2) \le \exp(C - \frac{R^2\tau_0^2}{Ch^{2(m-2N_1-n)}}), \tag{3.23}$$

which is $\mathcal{O}(1)\exp(-h^{-\delta})$ provided that

$$-2\widetilde{M} + 2\frac{\ln(1/\tau_0)}{\ln(1/h)} + 2(2N_1+n-m) \le -\delta. \tag{3.24}$$

Here $\tau_0 \le \sqrt{h}$ and if we choose $\tau_0 = \sqrt{h}$ or more generally bounded from below by some power of h, we see that (3.24) holds for any fixed δ, provided that m is sufficiently large.

In order to avoid such an extra assumption, we shall now represent α_j^0 for $h\mu_j^0 \le L$ as the sum of two independent Gaussian random variables. Let $j_0 = j_0(h)$ be the largest j for which $h\mu_j^0 \le L$. Put

$$\sigma' = \frac{1}{C} h^K e^{-Ch^{-\beta}},\ \text{where } K \ge \rho(M+1),\ C \gg 1 \tag{3.25}$$

so that $\sigma' \le \frac{1}{2}\sigma_j$ for $1 \le j \le j_0(h)$. The factor h^K is needed only when $\beta = 0$.

For $j \le j_0$, we may assume that $\alpha_j^0(\omega) = \alpha_j'(\omega) + \alpha_j''(\omega)$, where $\alpha_j' \sim \mathcal{N}(0,(\sigma')^2)$, $\alpha_j'' \sim \mathcal{N}(0,(\sigma_j'')^2)$ are independent random variables and

$$\sigma_j^2 = (\sigma')^2 + (\sigma_j'')^2,$$

so that

$$\sigma_j'' = \sqrt{\sigma_j^2 - (\sigma')^2} \asymp \sigma_j.$$

Put $q_\omega^1 = q_\omega' + q_\omega''$, where

$$q_\omega' = \sum_{h\mu_j^0\le L} \alpha_j'(\omega)\epsilon_j,\ q_\omega'' = \sum_{h\mu_j^0\le L} \alpha_j''(\omega)\epsilon_j.$$

Now (cf (3.20)) we write

$$P + h^m q^0_\omega = (P + h^m(q''_\omega + q^2_\omega)) + h^m q'_\omega.$$

The main result of [7] is valid for random perturbations of

$$P_0 := P + h^m(q''_\omega + q^2_\omega),$$

provided that $\|h^m(q''_\omega + q^2_\omega)\|_{H^s} \le h$, which again holds with a probability as in (3.22). The new random perturbation is now $h^m q'_\omega$ which we write as $\tau_0 h^{2N_1+n}\widetilde{q}_\omega$, where $\widetilde{q}_\omega$ takes the form

$$\widetilde{q}_\omega(x) = \sum_{0<h\mu^0_j\le L} \beta_j(\omega)\epsilon_j, \tag{3.26}$$

with new independent random variables

$$\beta_j = \frac{1}{\tau_0} h^{m-2N_1-n}\alpha'_j(\omega) \sim \mathcal{N}(0, (\frac{1}{\tau_0} h^{m-2N_1-n}\sigma'(h))^2). \tag{3.27}$$

Now, by (3.10),

$$\mathbf{P}(|\beta|^2_{\mathbf{C}^D} > R^2) \le \exp(\mathcal{O}(1)D - \frac{R^2\tau_0^2}{\mathcal{O}(1)(h^{m-2N_1-n}\sigma'(h))^2}).$$

Here by Weyl's law for the distribution of eigenvalues of elliptic self-adjoint differential operators, we have $D \asymp (L/h)^n$. Moreover, L, R behave like certain powers of h.

- In the case when $\beta = 0$, we choose $\tau_0 = h^{1/2}$. Then for any $a > 0$ we get
$$\mathbf{P}(|\beta|_{\mathbf{C}^D} > R) \le C\exp(-\frac{1}{Ch^a})$$
for any given fixed a, provided we choose K large enough in (3.25).
- In the case $\beta > 0$ we get the same conclusion with $\tau_0 = h^{-K}\sigma'$ if K is large enough.

In both cases, we see that the independent random variables β_j in (3.26), (3.27) have a joint probability density $C(h)e^{\Phi(\alpha;h)}L(d\alpha)$, satisfying (3.19) for some N_4 depending on K.

With $\kappa = 1/2$, we put

$$\epsilon_0(h) = h^\kappa((\ln\frac{1}{h})^2 + \ln\frac{1}{\tau_0}),$$

where τ_0 is chosen as above. Notice that $\epsilon_0(h)$ is of the order of magnitude $h^{\kappa-\beta}$ up to a power of $\ln\frac{1}{h}$. Then Theorem 1.1 in [7] gives:

Proposition 3.2 *There exists a constant $N_4 > 0$ depending on ρ, n, m such that the following holds: Let $\Gamma \Subset \dot{\mathbf{C}}$ have piecewise smooth boundary. Then $\exists C > 0$ such that for $0 < r \leq 1/C$, $\widetilde{\epsilon} \geq C\epsilon_0(h)$, we have with probability*

$$\geq 1 - \frac{C\epsilon_0(h)}{rh^{n+\max(n(M+1),N_4+\widetilde{M})}} e^{-\frac{\widetilde{\epsilon}}{C\epsilon_0(h)}} - Ce^{-\frac{1}{Ch}}, \tag{3.28}$$

that

$$|\#(h^m P_\omega^0) \cap \Gamma) - \frac{1}{(2\pi h)^n} \mathrm{vol}\,(p^{-1}(\Gamma))| \leq \tag{3.29}$$
$$\frac{C}{h^n}(\frac{\widetilde{\epsilon}}{r} + C(r + \ln(\frac{1}{r})\mathrm{vol}\,(p^{-1}(\partial\Gamma + D(0,r))))).$$

As noted in [6] this gives Weyl asymptotics provided that

$$\ln(\frac{1}{r})\mathrm{vol}\, p^{-1}(\partial\Gamma + D(0,r)) = \mathcal{O}(r^\alpha), \tag{3.30}$$

for some $\alpha \in]0,1]$ (which would automatically be the case if κ had been larger than 1/2 instead of being equal to 1/2), and we can then choose $r = \widetilde{\epsilon}^{1/(1+\alpha)}$, so that the right hand side of (3.29) becomes $\leq C\widetilde{\epsilon}^{\frac{\alpha}{1+\alpha}} h^{-n}$.

As in [6, 7] we also observe that if Γ belongs to a family $\mathcal{G}$ of domains satisfying the assumptions of the Proposition uniformly, then with probability

$$\geq 1 - \frac{C\epsilon_0(h)}{r^2 h^{n+\max(n(M+1),N_4+\widetilde{M})}} e^{-\frac{\widetilde{\epsilon}}{C\epsilon_0(h)}} - Ce^{-\frac{1}{Ch}}, \tag{3.31}$$

the estimate (3.29) holds uniformly and simultaneously for all $\Gamma \in \mathcal{G}$.

4 End of the proof

Let θ_1, θ_2, N_0 be as in Theorem 1.1, so that $P(\theta_1, N_0)$ and $P(\theta_2, N_0)$ hold. Combining the propositions 2.1, 2.2, 2.3, we see that (3.30) holds for every $\alpha < 1/N_0$ when $\Gamma = \Gamma^g_{\theta_1,\theta_2;1,\lambda}$, $\lambda > 0$ fixed, and Proposition 3.2 gives:

Proposition 4.1 *With the parameters as in Proposition 3.2 and for every $\alpha \in]0, \frac{1}{N_0}[$, we have with probability*

$$\geq 1 - \frac{C\epsilon_0(h)}{\widetilde{\epsilon}^{\frac{1}{1+\alpha}} h^{n+\max(n(M+1),N_4+\widetilde{M})}} e^{-\frac{\widetilde{\epsilon}}{C\epsilon_0(h)}} - Ce^{-\frac{1}{Ch}} \tag{4.1}$$

that

$$|\#(\sigma(h^m P_\omega) \cap \Gamma^g_{\theta_1,\theta_2;1,\lambda}) - \frac{1}{(2\pi h)^n} \mathrm{vol}\,(p^{-1}(\Gamma^g_{\theta_1,\theta_2;1,\lambda}))| \leq C\frac{\widetilde{\epsilon}^{\frac{\alpha}{1+\alpha}}}{h^n}. \tag{4.2}$$

Moreover, the conclusion (4.2) is valid simultaneously for all $\lambda \in [1,2]$ and all (θ_1,θ_2) in a set where $P(\theta_1,N_0)$, $P(\theta_2,N_0)$ hold uniformly, with probability

$$\geq 1 - \frac{C\epsilon_0(h)}{\widetilde{\epsilon}^{\frac{2}{1+\alpha}} h^{n+\max(n(M+1),N_4+\widetilde{M})}} e^{-\frac{\widetilde{\epsilon}}{C\epsilon_0(h)}} - Ce^{-\frac{1}{Ch}}. \tag{4.3}$$

For $0<\delta\ll 1$, choose $\widetilde{\epsilon}=h^{-\delta}\epsilon_0 \leq Ch^{\frac{1}{2}-\beta-\delta}(\ln\frac{1}{h})^2$, so that $\widetilde{\epsilon}/\epsilon_0 = h^{-\delta}$. Then for some N_5 we have for every $\alpha \in]0,1/N_0[$ that

$$|\#(\sigma(h^mP_\omega)\cap\Gamma^g_{\theta_1,\theta_2;1,\lambda}) - \frac{1}{(2\pi h)^n}\mathrm{vol}\,(p^{-1}(\Gamma^g_{\theta_1,\theta_2;1,\lambda}))| \leq \frac{C_\alpha}{h^n}(h^{\frac{1}{2}-\delta-\beta}(\ln\frac{1}{h})^2)^{\frac{\alpha}{1+\alpha}}, \tag{4.4}$$

simultaneously for $1\leq\lambda\leq 2$ and all (θ_1,θ_2) in a set where $P(\theta_1,N_0)$, $P(\theta_2,N_0)$ hold uniformly, with probability

$$\geq 1-\frac{C}{h^{N_5}}e^{-\frac{1}{Ch^\delta}}. \tag{4.5}$$

Here $\alpha/(1+\alpha)\nearrow 1/(N_0+1)$ when $\alpha\nearrow 1/N_0$, so the upper bound in (4.4) can be replaced by

$$\frac{C_\delta}{h^n}h^{(\frac{1}{2}-\beta-2\delta)/(N_0+1)}.$$

Assuming $P(\theta_1,N_0)$, $P(\theta_2,N_0)$, we want to count the number of eigenvalues of P_ω in

$$\Gamma_{1,\lambda}=\Gamma^g_{\theta_1,\theta_2;1,\lambda}$$

when $\lambda\to\infty$. Let $k(\lambda)$ be he largest integer k for which $2^k\leq\lambda$ and decompose

$$\Gamma_{1,\lambda}=(\bigcup_0^{k(\lambda)-1}\Gamma_{2^k,2^{k+1}})\cup\Gamma_{2^{k(\lambda)},\lambda}.$$

In order to count the eigenvalues of P^0_ω in $\Gamma_{2^k,2^{k+1}}$ we define h by $h^m2^k=1$, $h=2^{-k/m}$, so that

$$\begin{aligned}\#(\sigma(P^0_\omega)\cap\Gamma_{2^k,2^{k+1}}) &= \#(\sigma(h^mP^0_\omega)\cap\Gamma_{1,2}),\\ \frac{1}{(2\pi)^n}\mathrm{vol}\,(p^{-1}(\Gamma_{2^k,2^{k+1}})) &= \frac{1}{(2\pi h)^n}\mathrm{vol}\,(p^{-1}(\Gamma_{1,2})).\end{aligned}$$

Thus, with probability $\geq 1-C2^{\frac{N_5k}{m}}e^{-2^{\frac{\delta k}{m}}/C}$ we have

$$|\#(\sigma(P^0_\omega)\cap\Gamma_{2^k,2^{k+1}}) - \frac{1}{(2\pi)^n}\mathrm{vol}\,p^{-1}(\Gamma_{2^k,2^{k+1}})| \leq C_\delta 2^{\frac{kn}{m}}2^{-\frac{k}{m}(\frac{1}{2}-\beta-2\delta)\frac{1}{N_0+1}}. \tag{4.6}$$

Similarly, with probability $\geq 1 - C2^{N_5 k(\lambda)/m} e^{-2^{\delta k(\lambda)/m}/C}$, we have

$$|\#(\sigma(P_\omega^0) \cap \Gamma_{2^{k(\lambda)},\widetilde{\lambda}}) - \frac{1}{(2\pi)^n} \mathrm{vol}\, p^{-1}(\Gamma_{2^{k}(\lambda),\widetilde{\lambda}})| \leq C_\delta \lambda^{\frac{n}{m}} \lambda^{-\frac{1}{m}(\frac{1}{2}-\beta-2\delta)\frac{1}{N_0+1}}, \quad (4.7)$$

simultaneously for all $\widetilde{\lambda} \in [\lambda, 2\lambda[$.

Now, we proceed as in [1], using essentially the Borel–Cantelli lemma. Use that

$$\begin{aligned} \sum_{\ell}^{\infty} 2^{N_5 \frac{k}{m}} e^{-2^{\delta \frac{k}{m}}/C} &= \mathcal{O}(1) 2^{N_5 \frac{\ell}{m}} e^{-2^{\delta \frac{\ell}{m}}/C}, \\ \sum_{2^k \leq \lambda} 2^{k\frac{n}{m}} 2^{-\frac{k}{m}(\frac{1}{2}-\beta-2\delta)\frac{1}{N_0+1}} &= \mathcal{O}(1) \lambda^{\frac{n}{m}-\frac{1}{m}(\frac{1}{2}-\beta-2\delta)\frac{1}{N_0+1}}, \end{aligned}$$

to conclude that with probability $\geq 1 - C2^{N_5 \frac{\ell}{m}} e^{-2^{\delta \frac{\ell}{m}}/C}$, we have

$$|\#(\sigma(P_\omega^0) \cap \Gamma_{2^\ell,\lambda})| \leq C_\delta \lambda^{\frac{n}{m}-\frac{1}{m}(\frac{1}{2}-\beta-\delta)\frac{1}{N_0+1}} + C(\omega)$$

for all $\lambda \geq 2^\ell$. This statement implies Theorem 1.1. $\Box$

Proof of Theorem 1.2. This is just a minor modification of the proof of Theorem 1.1. Indeed, we already used the second part of Proposition 3.2, to get (4.7) with the probability indicated there. In that estimate we are free to vary (g, θ_1, θ_2) in $\mathcal{G}$ and the same holds for the estimate (4.6). With these modifications, the same proof gives Theorem 1.2. $\Box$

References

[1] W. Bordeaux Montrieux, *Loi de Weyl presque sûre et résolvante pour des opérateurs différentiels non-autoadjoints,* Thesis, CMLS, Ecole Polytechnique, 2008.

[2] E.B. Davies, *Semi-classical states for non-self-adjoint Schrödinger operators,* Comm. Math. Phys. 200(1)(1999), 35–41.

[3] M. Dimassi, J. Sjöstrand, *Spectral asymptotics in the semi-classical limit,* London Math. Soc. Lecture Notes Ser., 268, Cambridge Univ. Press, (1999).

[4] M. Hager, *Instabilité spectrale semiclassique d'opérateurs non-autoadjoints. II.* Ann. Henri Poincaré, 7(6)(2006), 1035–1064.

[5] M. Hager, J. Sjöstrand, *Eigenvalue asymptotics for randomly perturbed non-selfadjoint operators,* Math. Annalen, 342(1)(2008), 177–243.

[6] J. Sjöstrand, *Eigenvalue distribution for non-self-adjoint operators with small multiplicative random perturbations,* Ann. Fac. Sci. Toulouse, to appear.
http://arxiv.org/abs/0802.3584

[7] J. Sjöstrand *Eigenvalue distribution for non-self-adjoint operators on compact manifolds with small multiplicative random perturbations,*
http://arxiv.org/abs/0809.4182

[8] L.N. Trefethen, *Pseudospectra of linear operators,* SIAM Rev. 39(3)(1997), 383–406.

[9] M. Zworski, *A remark on a paper of E. B Davies: "Semi-classical states for non-self-adjoint Schrdinger operators"*, Proc. Amer. Math. Soc. 129(10)(2001), 2955–2957